

Exponential communication gap between weak and strong classical simulations of quantum communication

Alberto Montina

*Perimeter Institute for Theoretical Physics,
31 Caroline Street North, Waterloo,
Ontario N2L 2Y5, Canada*

and

*Facoltà di Informatica, Università della Svizzera Italiana,
Via G. Buffi 13, 6900, Lugano, Switzerland*

(Dated: January 16, 2013)

The most trivial way to simulate classically the communication of a quantum state is to transmit the classical description of the quantum state itself. However, this requires an infinite amount of classical communication if the simulation is exact. A more intriguing and potentially less demanding strategy would encode the full information about the quantum state into the probability distribution of the communicated variables, so that this information is never sent in each single shot. This kind of simulation is called weak, as opposed to strong simulations, where the quantum state is communicated in individual shots. In this paper, we introduce a bounded-error weak protocol for simulating the communication of an arbitrary number of qubits and a subsequent two-outcome measurement consisting of an arbitrary pure state projector and its complement. This protocol requires an amount of classical communication independent of the number of qubits and proportional to Δ^{-1} , where Δ is the error and a free parameter of the protocol. Conversely, a bounded-error strong protocol requires an amount of classical communication growing exponentially with the number of qubits for a fixed error. Our result improves a previous protocol, based on the Johnson-Lindenstrauss lemma, with communication cost scaling as $\Delta^{-2} \log \Delta^{-1}$.

I. INTRODUCTION

In the framework of quantum theory, the quantum state of a system does not represent any physical attribute of the system itself and just provides an operational (classical) description of a preparation procedure. This framework displays a dichotomy between the macroscopic realm, described through a classical language, and the quantum world, whose actual physical state is not provided by the formalism in terms of a classical picture. This difference of representation, which is the core of the measurement problem in quantum theory, has actually a simple solution. The quantum state, possibly supplemented by additional variables, can be interpreted as a physical classical state conditioning the outcomes of measurements. Such a solution of the measurement problem is employed, for example, by collapse theories and Bohm mechanics. This “cheap” way to introduce a classical language in the description of a quantum system is also the most trivial way to simulate classically quantum communication. Indeed a process of quantum state preparation, its transmission through a quantum channel and subsequent measurement can be classically simulated by directly transmitting the classical description of the quantum state itself. Classical theories promoting the quantum state to the rank of a physical variable are called ψ -ontic (in Greek, *ontos* means *that which is*).

In the recent years, there has been an increasing interest for an alternative class of hypothetical classical theories, called ψ -epistemic [1–15]. In their framework, the information about the quantum state is not stored in the classical state of each single quantum system, but

it is encoded in the probability distribution of the classical state (also called *ontic state*). In other words, in a ψ -epistemic theory, the quantum state represents statistical knowledge about the ontic state (in Greek, *episteme* means *knowledge*).

In quantum computer science, ψ -epistemic and ψ -ontic simulations are known as weak and strong simulations [16], respectively. The task of a strong simulation of a quantum measurement is to evaluate the outcome probabilities of every measurement outcomes with a possible bounded error. In other words, the task is to evaluate the quantum state after some processing. Conversely, a single shot of a weak simulation just generates a measurement outcome according to its quantum probability. A weak simulation is more similar to the actual experimental scenario that is simulated, where the final quantum state of a single system cannot be directly measured, but it is tomographically reconstructed through many repetitions of the same experimental procedure.

As discussed in Ref. [15], ψ -epistemic theories have an important role in quantum communication and are related to a very practical question: how many classical bits of communication are required to simulate exactly the communication of qubits? If only ψ -ontic simulations were feasible, then the communication cost would be trivially infinite, as a ψ -ontic simulation is carried out by communicating the full infinite information about the quantum state. Thus, it is clear that a protocol that classically simulates quantum communication through a finite amount of classical information (called, more concisely, finite communication protocol or FC protocol) is a kind of ψ -epistemic (weak) protocol. Furthermore, since

the mutual information, say I_m , between the quantum state and the ontic state is not greater than the communication cost, a FC protocol has I_m finite. In Ref. [15], we called ψ -epistemic models with finite I_m *completely ψ -epistemic*. Thus, a FC protocol is also completely ψ -epistemic. We proved that also the opposite is somehow true [15]. More precisely, we showed that a completely ψ -epistemic protocol can be turned into a FC protocol. The communication cost, say \mathcal{C} , of the derived protocol is essentially given by the mutual information I_m between the quantum state and the ontic state of the parent protocol. Indeed, using a recent result [17], we showed that

$$I_m \leq \mathcal{C} \leq I_m + 2 \log_2(I_m + 1) + 2 \log_2 e$$

(the second inequality can be strengthened under a suitable condition, as pointed out later). Furthermore, there is a procedure that turns parallel ψ -epistemic simulations into a global protocol with asymptotic communication cost, say \mathcal{C}^{asym} , per simulation exactly equal to the mutual information, as a consequence of the reversed Shannon theorem [18], in the form stated in Ref. [19].

Thus, the problem of finding a FC protocol is exactly equivalent to the problem of finding a completely ψ -epistemic protocol, since a FC protocol is completely ψ -epistemic and, moreover, a completely ψ -epistemic protocol can be always turned into a FC protocol. This procedure of turning a broader class of protocols into a subclass simplifies the task of deriving FC protocols, as point out in Ref. [15].

Completely ψ -epistemic models are known only for single qubits. Consequently, only FC protocols simulating the communication of single qubits are known. Toner and Bacon proved that the communication of two classical bits is sufficient to simulate the communication of a qubit [20]. In the case of many simulations performed in parallel, the asymptotic communication cost can be compressed to about 1.279 bits [15]. In this paper, we introduce a bounded-error ψ -epistemic protocol for simulating the communication of an arbitrary number of qubits, followed by a measurement consisting of an arbitrary pure state projector, say $|\phi\rangle\langle\phi|$, and its complement, $\mathbb{1} - |\phi\rangle\langle\phi|$. The error is a free parameter of the protocol and can be arbitrarily small. In the limit case of zero error, the model is ψ -ontic. Using the aforementioned procedure introduced in Ref. [15], we then derive an approximate FC protocol simulating the communication of n qubits. The amount of required classical communication is independent of n and inversely proportional to the worst-case error, say Δ , in the limit $n \gg \log \Delta^{-1}$. Conversely a bounded-error approximation of a brute force ψ -ontic simulation requires an amount of classical communication growing exponentially with the number of qubits for a fixed Δ . Our protocol improves a previous protocol, based on the Johnson-Lindenstrauss lemma, with communication cost scaling as $\Delta^{-2} \log \Delta^{-1}$ [21].

The paper is organized as follows. In section II, we introduce the general structure of a classical simulation of a quantum channel. We then define the communication

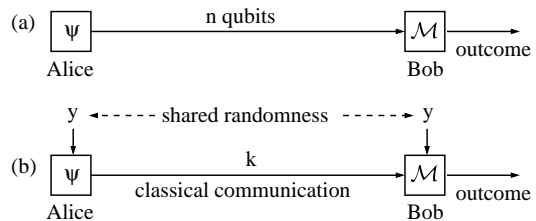


FIG. 1: (a) Schematic representation of a two-party scenario with one-way quantum communication. (b) Quantum communication is replaced by classical communication and the two parties are allowed to use shared randomness, y .

cost of the classical simulation and the communication complexity of a quantum channel. These definitions, as well as the definition of completely ψ -epistemic protocols, is slightly different from that given in Ref. [15]. In the previous definition, the communication cost was a function of the quantum state probability distribution, which needed to be specified. The new definition does not have this dependence. The definition of ψ -epistemic and ψ -ontic protocols is also generalized to the case of approximate simulations. In section III, the procedure of Ref. [15] is discussed with slight changes reflecting the different definition of communication cost. In Section IV, we use this procedure to derive the approximate FC protocol. Finally we compare the derived protocol with a bounded-error ψ -ontic protocol.

II. CLASSICAL SIMULATION OF QUANTUM CHANNELS

A classical simulation of a quantum channel simulates more correctly a process of preparation, transmission through the channel and measurement of a quantum state. We will consider only noiseless quantum channels. The scenario that is classically simulated is illustrated in Fig. 1a. A party, say Alice, prepares some qubits in a quantum state $|\psi\rangle$ and sends them to another party, say Bob. Bob then generates an outcome by performing some measurement $\mathcal{M} \equiv \{\hat{E}_1, \hat{E}_2 \dots\}$, where \hat{E}_i are positive semidefinite self-adjoint operators labeling events of the measurement \mathcal{M} . Notice that Alice has a classical description of the quantum state $|\psi\rangle$. In a more complicated scenario, which will not be discussed, Alice could perform some operations on qubits received from a third party.

A classical simulation of the two-party quantum scenario is illustrated in Fig. 1b. Instead of preparing the qubits in the state $|\psi\rangle$, Alice generates a classical variable k with a probability $\rho(k|y, \psi)$ depending on the quantum state and a possible random variable, y , shared with Bob. Thus, there is a mapping from the quantum state to a probability distribution of k ,

$$|\psi\rangle \xrightarrow{y} \rho(k|y, \psi). \quad (1)$$

The variable y is generated according to a probability distribution $\rho(y)$. The value of k is communicated by Alice to Bob. Finally, Bob generates an outcome \hat{E}_i with a probability $P(\hat{E}_i|k, y, \mathcal{M})$. The protocol simulates exactly the quantum channel if the probability of \hat{E}_i given $|\psi\rangle$ is equal to the quantum probability, that is, if

$$\int dk \int dy P(\hat{E}_i|k, y, \mathcal{M}) \rho(k|y, \psi) \rho(y) = \langle \psi | \hat{E}_i | \psi \rangle. \quad (2)$$

If the variable k takes a uncountably infinite number of values, the communication cost is infinite. Conversely, if k is a discrete variable, the communication cost, say \mathcal{C} , is defined as the maximum, over the space of probability distributions $\rho(\psi)$, of the Shannon entropy of the distribution $\rho(k|y)$ averaged over y , that is,

$$\mathcal{C} \equiv \max_{\rho(\psi)} \left\{ \int dy \rho(y) \left[- \sum_k \rho(k|y) \log_2 \rho(k|y) \right] \right\}. \quad (3)$$

This definition differs from that given in Ref. [15], where the cost was not maximized and was a function of the distribution $\rho(\psi)$, which needed to be specified. We define the *communication complexity* (denoted by \mathcal{C}_{min}) of a quantum channel as the minimal amount of classical communication \mathcal{C} required by an exact classical simulation of the quantum channel.

Shannon's source coding theorem [22] establishes an operational meaning of \mathcal{C} . Suppose that M independent simulations of M quantum channels are performed in parallel. Let k_i be the variable prepared with probability $\rho(k_i|y, \psi_i)$, where $|\psi_i\rangle$ is the quantum state prepared for the i -th quantum channel. Instead of communicating directly the variables k_i , we can encode them into a global k , so that the average number of communicated bits per simulation approaches \mathcal{C} as M goes to infinity. If the compression code is independent of the probability distribution $\rho(\psi)$, the compression rate is minimal, as stated by Shannon's theorem. In this parallelized protocol, the variables k_i are first generated according to the one-shot protocol and then compressed into the global variable k . However, it is possible to envisage a larger set of communication protocols where the global variable k is directly generated from the quantum states prepared in each single simulation. In other words, the probability distribution $\rho(k|y, \psi)$ of a single simulation is replaced by a probability distribution, say $\rho(k|y, \psi_1, \psi_2, \dots, \psi_M)$, depending on the whole set of M prepared quantum states. Thus, we have the mapping

$$\{|\psi_1\rangle, \dots, |\psi_M\rangle\} \xrightarrow{y} \rho(k|y, \psi_1, \psi_2, \dots, \psi_M), \quad (4)$$

which replaces the single-shot mapping (1). The asymptotic communication cost, \mathcal{C}^{asym} , is the cost of this parallelized simulation divided by M , for M going to infinity. We define the *asymptotic communication complexity*, \mathcal{C}_{min}^{asym} , of a quantum channel as the minimal asymptotic communication cost required for simulating the channel.

Since the set of protocols working for parallel simulations is larger than the set of protocols obtained by just compressing the communication of independent one-shot protocols, it is clear that

$$\mathcal{C}_{min}^{asym} \leq \mathcal{C}_{min}. \quad (5)$$

A classical channel $x_1 \rightarrow x_2$ from a stochastic variable x_1 to x_2 is defined by a conditional probability $\rho(x_1|x_2)$. The capacity of the channel is the maximum of the mutual information between x_1 and x_2 over the space of the input probability distributions $\rho(x_1)$ [22]. The mutual information of two variables x_1 and x_2 with probability distribution $\rho(x_1, x_2)$ is

$$I(X_1; X_2) = \sum_{x_1, x_2} \rho(x_1, x_2) \log_2 \frac{\rho(x_1, x_2)}{\rho(x_1)\rho(x_2)}.$$

Here, the capital letters refer to the stochastic variables, whereas their lower case refers to value taken by the variables. Whenever there is no ambiguity, we just use the lower case also for the stochastic variables themselves. From the chain rule

$$I(K, Y; \Psi) = I(Y; \Psi) + I(K; \Psi|Y) \quad (6)$$

and the fact that $|\psi\rangle$ and y are uncorrelated, we have that

$$I(K, Y; \Psi) = I(K; \Psi|Y). \quad (7)$$

The mutual information $I(K; \Psi|Y)$, for any $\rho(\psi)$, is smaller than or equal to the communication cost \mathcal{C} . Thus, from Eq. (7), we have that

$$\mathcal{C} \geq C(K, Y|\Psi), \quad (8)$$

where $C(K, Y|\Psi)$ is the capacity of the channel $|\psi\rangle \rightarrow \{k, y\}$ from the quantum state to the classical variables of the protocol.

As said in the introduction, an exact ψ -ontic protocol trivially simulates a quantum channel by sending the full infinite information about the quantum state. Conversely, in a ψ -epistemic theory, this information is encoded in the statistical distribution of the communicated variable. Thus, it is clear that an exact FC protocol is also a ψ -epistemic protocol. In the class of ψ -epistemic protocols, there is a interesting subclass of protocols that we call *completely ψ -epistemic* and characterized by the additional property that the channel capacity $C(K; Y|\Psi)$ is finite. This condition is slightly stronger than the finiteness of the mutual information $I(K; Y|\Psi)$ for some fixed $\rho(\psi)$, used in Ref. [15].

It is clear from Eq. (8) that a FC protocol is also a completely ψ -epistemic protocol. In the next section, we will show that also the opposite is somehow true. More precisely, we will show that there is procedure turning a completely ψ -epistemic protocol into a FC protocol. The hierarchy of the aforementioned classes is schematically represented in Fig. 2.

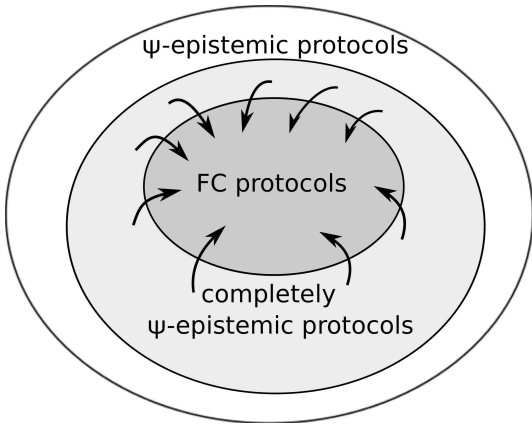


FIG. 2: In ψ -epistemic protocols, the full information about the quantum state is not communicated in each single shot, but it is encoded in statistical distribution of the communicated variable. Completely ψ -epistemic protocols are characterized by the additional property that the capacity of the channel $|\psi\rangle \rightarrow \{k, y\}$ is finite. There is a procedure that turns any completely ψ -epistemic protocol into a FC protocol. This procedure is represented by the arrows.

This hierarchy is broken in the case of bounded-error protocols. Indeed, a bounded-error ψ -ontic protocol can be also a FC protocol. Thus, a bounded-error FC protocol is not necessarily ψ -epistemic. We define an approximate ψ -ontic protocol as follows. First, Alice approximates the quantum state $|\psi\rangle$ with an element, say $|\psi_{sub}\rangle$, of a subset of vectors. It is important to stress that the mapping

$$|\psi\rangle \rightarrow |\psi_{sub}\rangle \quad (9)$$

does not depend on any stochastic variable, that is, there is a unique $|\psi_{sub}\rangle$ for each $|\psi\rangle$. Then, like in an exact ψ -ontic protocol, the full information about the $|\psi_{sub}\rangle$ is sent to the receiver. In this protocol, there is not an encoding of the quantum state information in the probability distribution of the communicated variable, this information is just partially erased to make it finite. By definition, an approximate ψ -epistemic protocol is any protocol that is not ψ -ontic. A bounded-error protocol generates the outcomes of the simulated measurement in accordance to the quantum probabilities with an error that is bounded by an arbitrarily small constant, which is a free parameter of the protocol. In the case of ψ -ontic simulations, this means that Bob has to receive, in a single shot, sufficient information so that he is able to evaluate the probability of any event with a bounded error.

III. FC PROTOCOLS FROM COMPLETELY ψ -EPISTEMIC PROTOCOLS

We now describe the procedure introduced in Ref. [15] for generating a FC protocol from a completely ψ -

epistemic protocol. This procedure is a consequence of the reverse Shannon theorem [18] and its one-shot version [17]. Given M copies of a channel $x \rightarrow y$, defined by the conditional probability $\rho(y|x)$ and with capacity C_{ch} , the reverse Shannon theorem states that they can be replaced by a global noiseless channel whose capacity is equal to $MC_{ch} + o(M)$, provided that the sender and receiver share some random variable. In other words the asymptotic communication cost of a parallel simulation of many copies of a channel $x \rightarrow y$ is equal to C_{ch} . A one-shot version of this theorem was recently reported in Ref. [17]. For independently simulated realizations, we have that

$$C_{ch} \leq \mathcal{C} \leq C_{ch} + 2\log_2(C_{ch} + 1) + 2\log_2 e. \quad (10)$$

Thus, the communication cost is C_{ch} plus a possible small additional cost that does not grow more than the logarithm of $C_{ch} + 1$. The second inequality was proved using an improved version of the rejection sampling method. If the probability distribution $\rho(y|x)$ and the distribution $\rho(y) = \sum_x \rho(y|x)\rho(x)$, obtained by maximizing the mutual information, are uniform in their support, it is possible to prove, using the standard rejection sampling method [23], the stronger constraint

$$C_{ch} \leq \mathcal{C} \leq C_{ch} + \log_2 e. \quad (11)$$

These results have an immediate application to the problem of deriving FC protocols from completely ψ -epistemic protocols. In general, a ψ -epistemic protocol can have an infinite communication cost. A strategy for making the amount of required communication finite and as small as possible is as follows. Instead of communicating directly the index k [see Eq. (1)], Alice can communicate an amount of information that allows Bob to generate k according to the probability distribution $\rho(k|y, \psi)$. By Eq. (10), the minimal amount of required communication is essentially equal to the capacity $C(K, Y|\Psi)$ of the channel $|\psi\rangle \rightarrow \{k, y\}$ (keep in mind that y and $|\psi\rangle$ are uncorrelated). If many simulations are performed in parallel, the reverse Shannon theorem implies that there is a classical simulation such that the asymptotic communication cost is strictly equal to $C(K, Y|\Psi)$.

In the following section, we will introduce a FC protocol working for measurements whose outcomes are an arbitrary projector $|\phi\rangle\langle\phi|$ and its complement $\mathbb{1} - |\phi\rangle\langle\phi|$. We will denote by $\mathcal{M}_2 \equiv \{|\phi\rangle\langle\phi|, \mathbb{1} - |\phi\rangle\langle\phi|\}$ this kind of two-outcome measurement.

IV. CLASSICAL PROTOCOL FOR MEASUREMENTS \mathcal{M}_2

In this section, we introduce an approximate completely ψ -epistemic model for simulating a quantum channel with a restriction on the set of measurements \mathcal{M} . More precisely, the model works for the two-outcome measurements $\mathcal{M}_2 = \{|\phi\rangle\langle\phi|, \mathbb{1} - |\phi\rangle\langle\phi|\}$, the outcomes

being an arbitrary rank-1 projector $|\phi\rangle\langle\phi|$ and its complement. Using the procedure described in the previous section, we will turn this protocol into a FC protocol, whose communication cost does not depend on the number of communicated qubits and is inversely proportional to the error. The completely ψ -epistemic model is a higher-dimensional generalization of the exact Kochen-Specker (KS) model working for single qubits. Thus, we first review this latter model and, then, we introduce its generalization.

A. Kochen-Specker model for single qubits

The Kochen-Specker model can be seen as a classical protocol for simulating the communication of single qubits. This model does not use a shared random variable y . The communicated variable is a vector, say $|x\rangle$, of the two-dimensional Hilbert space. Given the quantum state $|\psi\rangle$, the sender, say Alice, prepares $|x\rangle$ according to the probability distribution

$$\rho(x|\psi) = \frac{1}{2\pi^2} \left(|\langle x|\psi\rangle|^2 - \frac{1}{2} \right) \theta \left(|\langle x|\psi\rangle|^2 - \frac{1}{2} \right), \quad (12)$$

where θ is the Heaviside step function. She sends $|x\rangle$ to a second party, say Bob. The receiver then simulates a projective measurement with two outcomes denoted by a pair of orthogonal vectors, $|\phi\rangle$ and $|\phi_\perp\rangle$. He generates the outcome $|\phi\rangle$ with probability

$$P(\phi|x) = \theta \left(|\langle x|\phi\rangle|^2 - \frac{1}{2} \right) \quad (13)$$

This model simulates exactly a process of preparation, transmission and projective measurement of a qubit, that is,

$$\int d^2x P(\phi|x) \rho(x|\psi) = |\langle\phi|\psi\rangle|^2, \quad (14)$$

the right-hand side being the quantum probability of getting $|\phi\rangle$ given the quantum state $|\psi\rangle$.

In Ref. [27], we used this model to derive a protocol that classically simulates the communication of a single qubit by using 2 bits of classical communication. This model differs from that reported in Ref. [20]. Using the procedure described in the previous section, we also derived, from this model, a protocol working for simulations performed in parallel with asymptotic communication cost per simulation equal to about 1.28 bits [15], which is the capacity of the channel $|\psi\rangle \rightarrow |x\rangle$.

B. Higher-dimensional generalization of KS model

Now we present an approximate generalization of the KS model working in a higher-dimensional Hilbert space and for measurements \mathcal{M}_2 . Let us denote by N the

Hilbert space dimension. The simplest generalization of the KS model is as follows. The probability distribution $\rho(x|\psi)$, previously defined by Eq. (12), becomes

$$\rho(x|\psi) = R(|\langle x|\psi\rangle|^2) \theta(|\langle x|\psi\rangle|^2 - \cos^2 \theta_c), \quad (15)$$

where $R(\cdot)$ is a positive function such that probability distribution $\rho(x|\psi)$ is normalized and θ_c is a parameter in the interval $[0, \pi/2]$. The communicated variable $|x\rangle$ is now a vector of the N -dimensional Hilbert space. The conditional probability $P(\phi|x)$ defined by Eq. (13) takes now the more general form

$$P(\phi|x) = f(|\langle\phi|x\rangle|^2), \quad (16)$$

where $f(\cdot)$ is some function between zero and one. This model cannot simulate exactly the quantum scenario, unless $\theta_c \rightarrow 0$ (R going to infinite) and $f(y) = y$. In this limit case, the communicated variable is the quantum state itself, that is, Alice sends the quantum state to Bob, who uses it and the Born rule for evaluating the probability of the outcome $|\phi\rangle$. This model cannot be turned into a FC protocol, as the capacity of the channel $|\psi\rangle \rightarrow |x\rangle$ is infinite. Thus, we keep θ_c different from zero and choose $f(\cdot)$ and $R(\cdot)$ so that the error is as small as possible for any fixed θ_c . For a fixed θ_c , we will see that the error scales as N^{-1} .

Because of the phenomenon of the concentration of the measure in high dimension, there is a high probability that a vector $|x\rangle$ is generated close to the contour of the support of $\rho(x|\psi)$, that is, it is very likely that

$$|\langle x|\psi\rangle|^2 \simeq \cos^2 \theta_c. \quad (17)$$

Thus, for high-dimensional Hilbert spaces, the function $R(\cdot)$ can be approximated by a constant. Hereafter, for the sake of simplicity, we assume that $R(\cdot)$ is actually a constant, R_0 , determined by the normalization of $\rho(x|\psi)$,

$$R(|\langle x|\psi\rangle|^2) = R_0. \quad (18)$$

An estimate of the function $f(\cdot)$ is given by the following reasoning. By the concentration of the measure, it is possible to realize that, in high-dimensional Hilbert spaces, the vector $|x\rangle$ has also a high probability to be almost orthogonal to $|\phi\rangle - \langle\psi|\phi\rangle|\psi\rangle$. This property and Eq. (17) imply, that

$$|\langle\phi|x\rangle|^2 \simeq \cos^2 \theta_c |\langle\phi|\psi\rangle|^2. \quad (19)$$

By this equation and Eqs. (14,16), we infer that

$$f(y) \simeq \frac{y}{\cos^2 \theta_c}, \text{ for } 0 < y < \cos^2 \theta_c, \quad (20)$$

that is, the function $f(y)$ is very close to a linear function in the interval $[0; \cos^2 \theta_c]$. This heuristic reasoning suggests a trial function f of the form $f(y) = c_0 + c_1 y$, that is,

$$P(\phi|x) = c_0 + c_1 |\langle\phi|x\rangle|^2. \quad (21)$$

Initially, we assume that this linear form of $f(\cdot)$ holds in the whole domain $[0; 1]$ of the function and will show that Eq. (14) is exactly satisfied for a particular value of c_0 and c_1 . Thus, the model reproduces exactly quantum communication. However, this solution is not acceptable, since the conditional probability $P(\phi|x)$ turns out to be greater than 1 or negative for some vectors $|\phi\rangle$. This side effect is fixed by a slight change of the conditional probability $P(\phi|x)$. This correction introduces a small error, which will be evaluated.

The first step is to find the values of c_0 and c_1 such that Eq. (14) is exactly satisfied. From this equation and Eqs. (15,18,21), we have that

$$R_0 \int d^{2N-2}x (c_0 + c_1 |\langle x|\phi\rangle|^2) \theta(|\langle x|\psi\rangle|^2 - \cos^2 \theta_c) = |\langle \phi|\psi\rangle|^2. \quad (22)$$

Let us represent the vector $|x\rangle$ in the following coordinate system,

$$|x\rangle = \sin x_1 e^{iy_1} |\psi\rangle + \cos x_1 [e^{iy_2} \sin x_2 |1\rangle + \cos x_2 e^{iy_3} |w\rangle], \quad (23)$$

where $|1\rangle$ is a vector orthogonal to $|\psi\rangle$ and lying in the subspace spanned by $|\psi\rangle$ and $|\phi\rangle$, whereas $|w\rangle$ is any vector orthogonal to $|\psi\rangle$ and $|1\rangle$. The integration variables are $x_{1,2}$, $y_{1,2,3}$ and the $(N-2)$ -dimensional vector $|w\rangle$. The range of x_i and y_i is $[0; \pi/2]$ and $[0; 2\pi]$, respectively. In this coordinate system, the measure of an infinitesimal region is

$$d^2x d^3y d^{2N-5}w \sin x_1 \cos^{2N-3} x_1 \sin x_2 \cos^{2N-5} x_2. \quad (24)$$

Using this measure and performing the integral in Eq. (22), we find that,

$$c_0 + c_1 \left[\cos^2 \theta_c |\langle \phi|\psi\rangle|^2 + \frac{1}{N} \sin^2 \theta_c \right] = |\langle \phi|\psi\rangle|^2. \quad (25)$$

This equation is satisfied for any $|\psi\rangle$ and $|\phi\rangle$ if

$$\begin{aligned} c_0 &= -\frac{1}{N} \tan^2 \theta_c, \\ c_1 &= \cos^{-2} \theta_c. \end{aligned} \quad (26)$$

Thus, from Eq. (21) we have that

$$P(\phi|x) = \frac{|\langle x|\phi\rangle|^2}{\cos^2 \theta_c} - \frac{\tan^2 \theta_c}{N}, \quad (27)$$

which exhibits the correction term $N^{-1} \tan^2 \theta_c$ with respect to the heuristic estimate given by Eq. (20). Although $P(\phi|x)$ satisfies exactly Eq. (14), however it is greater than one for $|\langle x|\phi\rangle|^2 > \cos^2 \theta_c + N^{-1} \sin^2 \theta_c$ and negative for $|\langle x|\phi\rangle|^2 < N^{-1} \sin^2 \theta_c$. Thus, this function is not a conditional probability. The least invasive strategy for overcoming this problem is to set the function equal to one or zero where it is greater than one or negative, respectively. This makes the model approximate and we have to evaluate the introduced error. Summarizing, the approximate completely ψ -epistemic model that

generalizes the KS model is defined by the probability distributions

$$\rho(x|\psi) = R_0 \theta(|\langle x|\psi\rangle|^2 - \cos^2 \theta_c) \quad (28)$$

and

$$P(\phi|x) = \begin{cases} 1 & \text{for } |\langle x|\phi\rangle|^2 > \cos^2 \theta_c + \frac{\sin^2 \theta_c}{N} \\ 0 & \text{for } |\langle x|\phi\rangle|^2 < \frac{\sin^2 \theta_c}{N} \\ \frac{|\langle x|\phi\rangle|^2}{\cos^2 \theta_c} - \frac{\tan^2 \theta_c}{N} & \text{elsewhere} \end{cases} \quad (29)$$

Hereafter we will consider the most relevant parameter region given by the inequality

$$\tan^2 \theta_c < N. \quad (30)$$

This condition simplifies the error analysis and rules out only irrelevant protocols with error greater than $e^{-1} \simeq 0.36$.

Scrutinizing the exact quasi-probability distribution given by Eq. (27) and the approximate probability distribution, given by Eq. (29), it is easy to realize that the error of the protocol has two local maxima. One maximum, denoted by Δ_1 , is taken when $|\psi\rangle = |\phi\rangle$, the other one, say Δ_2 , when $|\psi\rangle$ and $|\phi\rangle$ are orthogonal.

Let us evaluate Δ_1 . It is given by

$$\Delta_1 = 1 - \int dx P(\psi|x) \rho(x|\psi). \quad (31)$$

Performing the integral, we find under constraint Eq.(30) that

$$\Delta_1 = \frac{1}{N} \left(1 - \frac{1}{N} \right)^N \tan^2 \theta_c \simeq \frac{\tan^2 \theta_c}{eN}. \quad (32)$$

The second local maximum is given by

$$\Delta_2 = \int dx P(\psi_\perp|x) \rho(x|\psi), \quad (33)$$

where $|\psi_\perp\rangle$ is any vector orthogonal to $|\psi\rangle$. For $\tan^2 \theta_c > (1 - N^{-1})^{-1}$, we find that

$$\Delta_2 = \Delta_1 - \frac{1}{N} \left(1 - \frac{1}{N} - \cot^2 \theta_c \right)^N \tan^2 \theta_c, \quad (34)$$

otherwise $\Delta_2 = \Delta_1$. Thus, Δ_1 is always greater than or equal to Δ_2 and the absolute maximum error, say Δ , is equal to Δ_1 ,

$$\Delta = \frac{1}{N} \left(1 - \frac{1}{N} \right)^N \tan^2 \theta_c \simeq \frac{1}{N} e^{-1} \tan^2 \theta_c. \quad (35)$$

C. ψ -epistemic FC protocol

The derived protocol is a completely ψ -epistemic model for $\theta_c \neq 0$, that is, the capacity of the channel

$|\psi\rangle \rightarrow |x\rangle$ is finite. Let us evaluate it. The mutual information $I(\Psi; X)$ is maximal for $\rho(\psi)$ constant. Since the distribution $\rho(x|\psi)$ is uniform where it is different to zero, it is easy to realize that the mutual information is the logarithm of the ratio between the volume of the space of unit vectors $|x\rangle$ and the volume of the support of $\rho(x|\psi)$, that is,

$$I(X; \Psi) = \log_2 \frac{\int dx \, 1}{\int dx \, \theta(|\langle x|\psi\rangle|^2 - \cos^2 \theta_c)}. \quad (36)$$

This equation gives

$$I(X; \Psi) = -2(N-1) \log_2 [\sin \theta_c]. \quad (37)$$

According to the procedure described in section III, there is FC protocol whose asymptotic communication cost, \mathcal{C}^{asym} , is the mutual information, thus $\mathcal{C}^{asym} = -2(N-1) \log_2 [\sin \theta_c]$. Using Eq. (35), we can express the communication cost as a function of the error Δ ,

$$\mathcal{C}^{asym} = (N-1) \log_2 \left[1 + \left(1 - \frac{1}{N} \right)^N \frac{1}{N\Delta} \right]. \quad (38)$$

Bearing in mind that the dimension N grows exponentially with the number of qubits, let us consider the relevant regime with $N \gg \Delta^{-1}$. In this limit, we have that

$$\mathcal{C}^{asym} \simeq \frac{1}{e \log_2 e} \Delta^{-1} \simeq \frac{0.255}{\Delta}. \quad (39)$$

Thus, the communication cost turns out to be independent of the number of qubits and inversely proportional to the error in the high-dimensional limit.

For single-shot simulations, the communication cost is bounded by Ineqs. (11), since the distributions $\rho(x|\psi)$ and $\rho(x)$ are uniform in their support. Thus, the single-shot communication cost is equal to \mathcal{C}^{asym} plus a possible additional cost that is not greater than $\log_2 e \simeq 1.443$.

1. Alternative protocol

An alternative ψ -epistemic FC protocol can be derived using a dimensional reduction strategy [21]. Since this strategy is quite known in quantum cryptography and the resulting protocol has a worse communication cost with respect to the previous result, we just give a brief presentation of this protocol. The protocol is as follows. Alice and Bob share a random unitary operator, say \hat{U} . Alice evaluates the normalized vector

$$|\psi_t\rangle \equiv \frac{\hat{P}\hat{U}|\psi\rangle}{\|\hat{P}\hat{U}|\psi\rangle\|}, \quad (40)$$

where \hat{P} is an operator projecting into a subspace with dimension N_s . Similarly, Bob evaluates the vector

$$|\phi_t\rangle \equiv \frac{\hat{P}\hat{U}|\phi\rangle}{\|\hat{P}\hat{U}|\phi\rangle\|}. \quad (41)$$

Alice approximates the vector $|\psi_t\rangle$ with a vector $|\psi_{net}\rangle$ of an ϵ -net [24]. An ϵ -net is a set of vectors such that each vector of the Hilbert space is within the distance ϵ of some vector in the set. Let us denote by M the number of vector of the ϵ -net. There is an ϵ -net such that

$$M \propto \left(\frac{5}{\epsilon} \right)^{2N_s}, \quad (42)$$

as proved in Ref. [25]. Alice then sends $|\psi_{net}\rangle$ to Bob. This requires an amount of communication equal to

$$\mathcal{C} = \log_2 M. \quad (43)$$

Finally, Bob generates the outcome $|\phi\rangle\langle\phi|$ with probability equal to $|\langle\psi_{net}|\phi_t\rangle|^2$. This protocol is an approximate simulation of the quantum scenario. There are two sources of error. The first one is the subspace projection. It introduces an error, say Δ_{proj} proportional to $N_s^{-1/2}$ and independent of the dimension N of the original Hilbert space [26],

$$\Delta_{proj} \propto N_s^{-1/2}. \quad (44)$$

The second source is the ϵ -net, which introduces an error, Δ_{net} , proportional to ϵ , so that Eq. (42) can be written as

$$M \propto \left(\frac{\alpha}{\Delta_{net}} \right)^{2N_s}, \quad (45)$$

α being some constant. From Eq. (43,44,45) we find that the communication cost, as a function of Δ_{net} and Δ_{proj} , is

$$\mathcal{C} \simeq \frac{\beta}{\Delta_{proj}^2} \log_2 \frac{\alpha}{\Delta_{net}}, \quad (46)$$

where β is a constant. This model is less efficient than the previously derived model. Indeed, the communication cost grows as $\Delta^{-2} \log_2 \Delta^{-1}$, whereas the amount of communication in the previous model is proportional to Δ^{-1} [see Eq. (39)].

D. Approximate ψ -ontic (strong) simulation

In Sec. IV C, we have presented a bounded-error ψ -epistemic protocol whose communication cost is independent of the number of qubits and inversely proportional to the worst-case error. Now, we compare this protocol with a bounded-error ψ -ontic protocol working for the same quantum scenario and show that the latter requires an amount of classical communication growing exponentially with the number of qubits for a fixed worst-case error.

In an exact ψ -ontic simulation, Alice sends the classical description of the quantum state $|\psi\rangle$ to Bob. In other terms, Bob receives sufficient information to evaluate the

probability $|\langle\phi|\psi\rangle|^2$ of any arbitrary event $|\phi\rangle\langle\phi|$. As defined in Sec. II, in a bounded-error ψ -ontic protocol, Alice has to send an estimate, say $|\psi_e\rangle$, of the quantum state so that Bob can evaluate the probability of any event with an error bounded by a constant, say Δ . This kind of simulation is also called strong simulation [16]. The vector $|\psi_e\rangle$ is chosen in an ϵ -net of vectors so that the distance between $|\psi_e\rangle$ and $|\psi\rangle$ cannot be bigger than about Δ . The number of elements of the ϵ -net, say M , scales exponentially with N . More precisely,

$$M \propto \left(\frac{\alpha}{\Delta}\right)^{2N}. \quad (47)$$

This scale law is optimal. Thus, the communication cost of the bounded-error protocol is

$$\mathcal{C} \simeq 2N \log_2 \frac{\alpha}{\Delta} \quad (48)$$

It scales linearly with the Hilbert space dimension, that is, exponentially with the number of qubits for a fixed worst-case error. Thus, there is an exponential gap between the communication cost of the bounded-error ψ -epistemic protocol and the bounded-error ψ -ontic protocol.

V. CONCLUSION

There are two possible ways to simulate classically a quantum channel. In the trivial way, the full classical description of the quantum state is communicated by the sender to the receiver. In other words, the receiver gets, in a single shot, the full information about the probabilities of every event of any measurement. This simulation, called ψ -ontic, requires an infinite amount of communication. In the second way, the information about the quantum state is encoded in the probability distribution of the communicated variable. The receiver gets an amount of information that is sufficient to generate

an event according to the quantum probabilities, but he does not get the information about the quantum probabilities themselves. We have called this kind of protocol ψ -epistemic. In quantum computer science, they are also known as weak simulations.

In this paper, we have presented a bounded-error ψ -epistemic protocol that classically simulates the communication of an arbitrary number n of qubits with subsequent measurement consisting of an arbitrary pure state projector and its complement. The communication cost is independent of n and inversely proportional to the worst-case error Δ in the limit $n \gg \Delta$. Conversely, a bounded-error ψ -ontic protocol requires an amount of classical communication growing exponentially with the number of qubits for a fixed error. Our model beats a previous protocol based on the Johnson-Lindenstrauss lemma, whose communication cost scales as $\Delta^{-2} \log_2 \Delta^{-1}$, Δ being the error [21]. The purpose of this work is to provide a further illustration that ψ -epistemic (weak) simulations of quantum systems can be more effective than ψ -ontic (strong) simulations. The still open challenge is to find exact completely ψ -epistemic theories of quantum systems or, equivalently, exact FC protocols, whose existence is still debated. The state of the art about FC protocols is the lower bound $2^n - 1$ for the communication cost of a noiseless quantum channel with capacity n [28]. In a following paper [29], we will introduce a constructive procedure to evaluate the communication complexity of general quantum channels. This procedure is based on the procedure used here, which was introduced in Ref. [15] and reviewed in Sec. III.

Acknowledgments. The author acknowledge useful discussions with Fernando Brandao. Research at Perimeter Institute for Theoretical Physics is supported in part by the Government of Canada through NSERC and by the Province of Ontario through MRI. This work is partially supported by the Swiss National Science Foundation, the NCCR QSIT, and the COST action on Fundamental Problems in Quantum Physics.

-
- [1] L. Hardy, Stud. Hist. Phil. Sci. B **35**, 267 (2004).
 - [2] A. Montina, Phys. Rev. Lett. **97**, 180401 (2006).
 - [3] R. W. Spekkens, Phys. Rev. A **75**, 032110 (2007).
 - [4] A. Montina, Phys. Rev. A **77**, 022104 (2008);
 - [5] N. Harrigan, R. W. Spekkens, Found. Phys. **40**, 125 (2010).
 - [6] A. Montina, Phys. Rev. A **83**, 032107 (2011).
 - [7] A. Montina, Phys. Lett. A **375**, 1385 (2011)
 - [8] S. D. Bartlett, T. Rudolph, R. W. Spekkens, Phys. Rev. A **86**, 012103 (2012).
 - [9] A. Montina, Phys. Rev. Lett. **108**, 160501 (2012).
 - [10] M. F. Pusey, J. Barrett, T. Rudolph, Nature Physics, **8**, 476 (2012).
 - [11] P. G. Lewis, D. Jennings, J. Barrett, T. Rudolph, Phys. Rev. Lett. **109**, 150404 (2012).
 - [12] R. Colbeck, R. Renner, Phys. Rev. Lett. **108**, 150402 (2012).
 - [13] M. Schlosshauer, A. Fine, Phys. Rev. Lett. **108**, 260404 (2012).
 - [14] L. Hardy, arXiv:1205.1439.
 - [15] A. Montina, Phys. Rev. Lett. **109**, 110501 (2012).
 - [16] R. Jozsa, A. Miyake, Proc. R. Soc. A **464**, 3089 (2008); M. Van den Nest, Quant. Inf. Comp. **10**, 258 (2010); M. Van den Nest, Quant. Inf. Comp. **11**, 784 (2011).
 - [17] P. Harsha, R. Jain, D. McAllester, J. Radhakrishnan, IEEE Trans. Inf. Theory **56**, 438 (2010).
 - [18] C. H. Bennett, P. Shor, J. Smolin, and A. V. Thapliyal, IEEE Trans. Inf. Theory **48**, 2637 (2002).
 - [19] A. Winter, arXiv:quant-ph/0208131.
 - [20] B. F. Toner, D. Bacon, Phys. Rev. Lett. **91**, 187904 (2003).
 - [21] Fernando Brandao, private communication.

- [22] T. M. Cover and J. A. Thomas, *Elements of Information Theory* (Wiley, New York, 1991).
- [23] W. H. Press, S. A. Teukolsky, W. T. Vetterling, B. P. Flannery, *Numerical Recipes: The Art of Scientific Computing* (Cambridge University Press, New York, 2007); L. Devroye, *Non-Uniform Random Variate Generation* (Springer-Verlag, New York, 1986).
- [24] W. A. Sutherland, *Introduction to metric and topological spaces* (Oxford University Press, Oxford, 1975).
- [25] P. Hayden, D. Leung, P. W. Shor, A. Winter, Commun. Math. Phys. **250**, 371 (2004).
- [26] W. Johnson and J. Lindenstrauss, Contemporary Mathematics **26**, 189 (1984).
- [27] A. Montina, Phys. Rev. A **84**, 042307 (2011).
- [28] A. Montina, Phys. Rev. A **84**, 060303(R) (2011).
- [29] A. Montina, M. Pfaffhauser, S. Wolf, to be published.